SEMESTRAL EXAMINATION

COMPLEX ANALYSIS
B. MATH III YEAR
I SEMESTER, 2008-2009

Notations: $U = \{z : |z| < 1\}, H(\Omega)$ is the space of holomorphic functions on the region $\Omega, C(X)$ is the space of continuous functions on the compact space X, γ^* is the range of the path γ.

1. Does there exist a non-constant entire function f such that $\left|f(z^3)\right| \leq 1 + |z|$ for all z? [15]

2. Prove that if $\gamma : [0,1] \to \mathbb{C}$ is a continuously differentiable then $f(z) = \int_\gamma \frac{g(\zeta)}{\zeta - z} d\zeta$ defines a holomorphic function on $\mathbb{C} \backslash \gamma^*$ for any continuous function g on γ^*. [15]

3. If $f \in C(\bar{U}) \cap H(U)$ and $|f(z) - 1 - 2z| < 1$ for $|z| = 1$ prove that f has a unique zero in U. [10]

4. Let $z_n \in \mathbb{C} \backslash \{0\}$ for all n. Prove that $\prod_{n=1}^{\infty} z_n$ converges if and only if $\sum_{n=1}^{\infty} Log(z_n)$ converges. [15]

5. Let f and g be entire functions, $\epsilon, \Delta \in (0, \infty)$ and $1 \leq |f(z)| \leq |g(z)| \, |z|^{-1-\epsilon}$ for $|z| \geq \Delta$. Prove that the sum of the residues of $\frac{f}{g}$ at all its poles is 0. [15]

6. Let $\Omega = \{z : \text{Re}(z) > 0\}$. Give an example of a bijection from Ω onto U which is bi-holomorphic. Is it possible to find a continuous bijection from $\bar{\Omega}$ onto $\bar{U}$ which is holomorphic in Ω and maps Ω onto U? [12+3]

7. Evaluate $\int_\gamma \frac{3z^3+2}{(z-1)(z^2+9)} dz$ where $\gamma(t) = 4e^{2\pi i t} (0 \leq t \leq 1)$. [10]

8. Evaluate $\int_0^\infty \frac{x^2}{x^6+1} dx$ by the method of residues. [Exact value need not be computed] [15]